

The Warped Product of Hamiltonian Spaces

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Abstract

In this work, the warped product of Hamilton spaces is introduced and it is shown that these spaces obtain Hamiltonian structure as well. Then, the geometric properties of warped product Hamilton spaces such as their nonlinear connections and natural cotangent bundle structures are studied. Moreover, we prove some theorems that show geometric relation between warped product Hamiltonian space and its base Hamiltonian manifolds. For example, we prove that for non-constant warped function f , the sasaki lifted metric G of Hamiltonian warped product space is bundle-like for its vertical foliation if and only if based Hamiltonian spaces are Riemannian manifolds.

Keywords: Warped Product, Hamilton Space, Bundle-like metric.
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1 Introduction

The notion of warped product spaces was introduced to study the manifolds with negative curvatures by Bishop and O'Neill c.f. [6]. Afterwards, warped product was used to model the standard space time, especially in the neighborhood of stars and black holes [15]. However warped product spaces was developed in Riemannian manifolds enormously (see, [8, 5]), the warped product of Finsler manifolds was introduced in 2001 by Kozma [9] and recently developed by one of the present authors c.f. [1, 7, 17]. In this work, warped product of Hamilton spaces is introduced and it is shown that these spaces obtain Hamiltonian structure as well. Then, some geometric properties of warped product Hamilton spaces are studied.

Lagrange space has been certified as an excellent model for some important problems in Relativity, Gauge Theory and Electromagnetism [10, 11]. The geometry of Lagrange spaces gives a model for both the gravitational

and electromagnetic field. Moreover, this structure plays a fundamental role in study of the geometry of tangent bundle TM . The geometry of cotangent bundle T^*M and tangent bundle TM which follows the same outlines are related by Legendre transformation. From this duality, the geometry of a Hamilton space can be obtained from that of certain Lagrange space and vice versa. Using this duality several important results in the Hamiltonian spaces can be obtained: the canonical nonlinear connection, the canonical metrical connection etc. Therefore, the theory of Hamilton spaces has the same symmetry and beauty like Lagrange geometry. Moreover, it gives a geometrical framework for the Hamiltonian theory of Mechanics or Physical fields. With respect to the importance of these spaces in Physical areas, present work is formed to develop the concept of warped product on Hamiltonian spaces. Aiming at our purpose, this paper is organized in the following way.

Let (M, H) be the warped Hamilton space of Hamiltonian spaces (M_1, H_1) and (M_2, H_2) . In section 2, the notion of warped product Hamiltonian spaces is presented and some natural geometrical properties of cotangent bundle for a warped manifold are given. In section 3, it is shown that (M, H) is a Hamilton space and its canonical nonlinear connections are calculated as well. Moreover, sasakian lifted metric G on T^*M is introduced. In section 4, the Levi-Civita connection of Riemannian metric G on T^*M is calculated. Finally in section 5, we prove some theorems that they show close relation between geometry of warped product Hamilton manifolds and their base Hamiltonian spaces.

2 Preliminaries and Notations

Let $\mathbb{H}_1^n = (M_1, H_1)$ and $\mathbb{H}_2^m = (M_2, H_2)$ be two Hamilton spaces with $\dim \mathbb{H}_1^n = n$ and $\dim \mathbb{H}_2^m = m$, respectively. The warped product of these spaces is denoted by $\mathbb{H} = (M, H)$ where:

$$M = M_1 \times M_2 \quad \text{and} \quad H = H_1 + fH_2 \quad (1)$$

for some smooth function $f : M_1 \rightarrow \mathbb{R}^+$. Then a coordinate system in M is denoted by $\{(U \times V, \varphi \times \psi)\}$, where $\{(U, \varphi)\}$ and $\{(V, \psi)\}$ are coordinate systems in M_1 and M_2 , respectively, such that each $\mathbf{x} = (x, z) \in M$ has local expression (x^i, z^α) . It is notable that throughout the paper, the indices $\{i, j, k, \dots\}$ and $\{\alpha, \beta, \lambda, \dots\}$ are used for the ranges $1, \dots, n$ and $1, \dots, m$, respectively. Let canonical projections of T^*M_1 on M_1 and T^*M_2 on M_2 be denoted by π_1 and π_2 , respectively. The fibres of cotangent bundle at $\mathbf{x} = (x, z) \in M$ is $T_{(x,z)}^*M = T_x^*M_1 \oplus T_z^*M_2$, therefore $T^*M = T^*M_1 \oplus T^*M_2$.

The induced coordinate system on T^*M_1 and T^*M_2 are (x^i, p_i) and (z^α, q_α) , respectively, which the coordinate p_i and q_α are called *momentum variables* [12]. The change of these coordinates on T^*M_1 and T^*M_2 are given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n), \\ \text{rank} \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right) = n, \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j \end{cases} \quad \begin{cases} \tilde{z}^\alpha = \tilde{z}^\alpha(z^1, \dots, z^m), \\ \text{rank} \left(\frac{\partial \tilde{z}^\alpha}{\partial z^\beta} \right) = m, \\ \tilde{q}_\alpha = \frac{\partial z^\beta}{\partial \tilde{z}^\alpha} q_\beta \end{cases} \quad (2)$$

Let $(\mathbf{x}, \mathbf{p}) = (x, z, p, q) \in T^*M = T^*M_1 \oplus T^*M_2$, the tangent space at (\mathbf{x}, \mathbf{p}) to T^*M is denoted by $T_{(\mathbf{x}, \mathbf{p})}T^*M$ that is a $2(n+m)$ -dimensional vector space. The natural basis induced on $T_{(\mathbf{x}, \mathbf{p})}T^*M$ by local coordinate of T^*M_1 and T^*M_2 is $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_\alpha}\}$. These coordinates are changed with respect to transformations (2) as follows:

$$\begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}_j}{\partial x^i} \frac{\partial}{\partial \tilde{p}_j} \\ \frac{\partial}{\partial z^\alpha} = \frac{\partial \tilde{z}^\beta}{\partial z^\alpha} \frac{\partial}{\partial \tilde{z}^\beta} + \frac{\partial \tilde{q}_\beta}{\partial z^\alpha} \frac{\partial}{\partial \tilde{q}_\beta} \\ \frac{\partial}{\partial p_i} = \frac{\partial x^j}{\partial p_i} \frac{\partial}{\partial \tilde{x}^j} \\ \frac{\partial}{\partial q_\alpha} = \frac{\partial z^\beta}{\partial q_\alpha} \frac{\partial}{\partial \tilde{z}^\beta} \end{cases} \quad (3)$$

In this work, the notations $\dot{\partial}^i$ and $\dot{\partial}^\alpha$ are used instead of $\frac{\partial}{\partial p_i}$ and $\frac{\partial}{\partial q_\alpha}$, respectively, similar to the notations in [12]. The Jacobian matrix of transformations (3) is

$$Jac := \begin{pmatrix} \frac{\partial \tilde{x}^j}{\partial x^i} & 0 & 0 & 0 \\ 0 & \frac{\partial \tilde{z}^\beta}{\partial z^\alpha} & 0 & 0 \\ \frac{\partial \tilde{p}_j}{\partial x^i} & 0 & \frac{\partial x^j}{\partial \tilde{x}^j} & 0 \\ 0 & \frac{\partial \tilde{q}_\beta}{\partial z^\alpha} & 0 & \frac{\partial z^\beta}{\partial \tilde{z}^\beta} \end{pmatrix} \quad (4)$$

It follows

$$\det Jac = 1$$

By means of last equation, we have the following corollary.

Corollary 2.1. *The manifold $T^*M = T^*M_1 \oplus T^*M_2$ is orientable.*

Let $\bar{\partial}^a$ and $\frac{\partial}{\partial \mathbf{x}^a}$ be abbreviations for $\dot{\partial}^i \delta_i^a + \dot{\partial}^\alpha \delta_{\alpha+n}^a$ and $\frac{\partial}{\partial x^i} \delta_a^i + \frac{\partial}{\partial z^\alpha} \delta_{\alpha+n}^a$, respectively, where the indices $\{a, b, c, \dots\}$ are used for the range $1, \dots, n +$

m . Throughout the paper, these notations and range of the indices are established.

we know that there are some natural structures live on the cotangent bundle T^*M , it would be interesting to present them on cotangent bundle of a warped product Hamilton space. First, the *Liouville-Hamilton vector field* of T^*M is given by:

$$C^* := \mathbf{p}_a \bar{\partial}^a = p_i \dot{\partial}^i + q_\alpha \dot{\partial}^\alpha = C_1^* + C_2^* \quad (5)$$

where, C_1^* and C_2^* denote the Liouville-Hamilton vector fields of T^*M_1 and T^*M_2 , respectively.

Next, the *Liouville 1-form* θ on T^*M is defined by:

$$\theta := \mathbf{p}_a d\mathbf{x}^a = p_i dx^i + q_\alpha dz^\alpha = \theta_1 + \theta_2 \quad (6)$$

where, θ_1 and θ_2 are liouville 1-forms of T^*M_1 and T^*M_2 , respectively.

And, the *canonical symplectic structure* ω on T^*M is defined by $\omega = d\theta$ and has local expression as follows:

$$\omega := d\mathbf{p}_a \wedge d\mathbf{x}^a = dp_i \wedge dx^i + dq_\alpha \wedge dz^\alpha = \omega_1 + \omega_2 \quad (7)$$

where ω_1 and ω_2 are canonical symplectic structures of T^*M_1 and T^*M_2 , respectively.

Finally, if the Poisson bracket on cotangent bundle of T^*M_1 , T^*M_2 and T^*M are denoted by $\{.,.\}_1$, $\{.,.\}_2$ and $\{.,.\}$, respectively, then they are related as follows:

$$\{f, h\} = \bar{\partial}^a f \frac{\partial h}{\partial \mathbf{x}^a} - \bar{\partial}^a h \frac{\partial f}{\partial \mathbf{x}^a} = \{f, h\}_1 + \{f, h\}_2 \quad (8)$$

where $f, h \in C^\infty(T^*M)$.

The *Hamilton vector field* of Hamiltonian function H is denoted by X_H and satisfied the following equation

$$\iota_{X_H} \omega = -dH$$

Let X_{H_1} and X_{H_2} be Hamilton vector fields of the spaces \mathbb{H}_1^n and \mathbb{H}_2^m , respectively, then the following theorem gives an expression of X_H .

Theorem 2.1. *Suppose that $\mathbb{H} = (M, H)$ is warped product Hamilton space defined in (1). Then the Hamilton vector field of \mathbb{H} is given by:*

$$X_H = X_{H_1} + f X_{H_2} - H_2 \frac{\partial f}{\partial x^i} \dot{\partial}^i$$

Proof. By definition of Hamilton vector fields i.e. $\iota_{X_H} \omega = -dH$. It is a straightforward calculation to complete the prove of corollary. \square

3 Nonlinear Connection on Warped Product Hamilton Space

For Hamiltonian spaces \mathbb{H}_1^n and \mathbb{H}_2^m , the followings

$$\begin{cases} g^{ij} = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j H_1 \\ g^{\alpha\beta} = \frac{1}{2} \dot{\partial}^\alpha \dot{\partial}^\beta H_2 \end{cases} \quad (9)$$

define metric tensors of spaces \mathbb{H}_1^n and \mathbb{H}_2^m , respectively. The metric tensor of warped product Hamiltonian space (M, H) is given by:

$$(g^{ab}) = \begin{pmatrix} \frac{1}{2} \bar{\partial}^a \bar{\partial}^b H \\ 0 \\ 0 & f g^{\alpha\beta} \end{pmatrix} \quad (10)$$

Now, it is easy to check that (M, H) is a Hamilton space as well. In [12], the canonical nonlinear connections of a Hamilton space were presented. By means of that definition, we obtain canonical nonlinear connections of \mathbb{H}_1^n , \mathbb{H}_2^m and \mathbb{H} , respectively, as follows

$$\begin{cases} N_{ij} = \frac{1}{4} \{g_{ij}, H_1\} - \frac{1}{4} \left(g_{ik} \frac{\partial^2 H_1}{\partial p_k \partial x^j} + g_{jk} \frac{\partial^2 H_1}{\partial p_k \partial x^i} \right) \\ N_{\alpha\beta} = \frac{1}{4} \{g_{\alpha\beta}, H_2\} - \frac{1}{4} \left(g_{\alpha\gamma} \frac{\partial^2 H_2}{\partial q_\gamma \partial z^\beta} + g_{\beta\gamma} \frac{\partial^2 H_2}{\partial q_\gamma \partial z^\alpha} \right) \\ \bar{N}_{ab} = \frac{1}{4} \{g_{ab}, H\} - \frac{1}{4} \left(g_{ac} \frac{\partial^2 H}{\partial \mathbf{p}_c \partial \mathbf{x}^b} + g_{bc} \frac{\partial^2 H}{\partial \mathbf{p}_c \partial \mathbf{x}^a} \right) \end{cases} \quad (11)$$

where (g_{ij}) , $(g_{\alpha\beta})$ and (g_{ab}) are the inverse matrixes of (g^{ij}) , $(g^{\alpha\beta})$ and (g^{ab}) , respectively. Then the nonlinear connections \bar{N}_{ab} of Hamiltonian space \mathbb{H} are related to the those of \mathbb{H}_1^n and \mathbb{H}_2^m as follows

$$\begin{cases} \bar{N}_{ij} = N_{ij} + \frac{1}{4} \dot{\partial}^k g_{ij} \frac{\partial f}{\partial x^k} H_2 \\ \bar{N}_{\alpha\beta} := \bar{N}_{(\alpha+n)(\beta+n)} = N_{\alpha\beta} - \frac{1}{4f^2} g_{\alpha\beta} \dot{\partial}^k H_1 \frac{\partial f}{\partial x^k} \\ \bar{N}_{i\alpha} := \bar{N}_{i(\alpha+n)} = -\frac{1}{4f} g_{\alpha\beta} \dot{\partial}^\beta H_2 \frac{\partial f}{\partial x^i} \end{cases} \quad (12)$$

The kernel of π_* where

$$\pi := (\pi_1, \pi_2) : T^*M_1 \oplus T^*M_2 \longrightarrow M_1 \times M_2$$

is known as vertical bundle on T^*M and denoted by VT^*M . Locally ΓVT^*M is spanned by

$$\left\{ \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}, \frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_m} \right\}$$

Using the functions \bar{N}_{ij} , $\bar{N}_{i\alpha}$ and $\bar{N}_{\alpha\beta}$ the nonholomorphic vector fields are defined by

$$\begin{cases} \frac{\delta^*}{\delta^* x^i} := \frac{\delta^*}{\delta^* \mathbf{x}^i} = \frac{\partial}{\partial x^i} + \bar{N}_{ij} \dot{\partial}^j + \bar{N}_{i\alpha} \dot{\partial}^\alpha \\ \frac{\delta^*}{\delta^* z^\alpha} := \frac{\delta^*}{\delta^* \mathbf{x}^{\alpha+n}} = \frac{\partial}{\partial z^\alpha} + \bar{N}_{\alpha i} \dot{\partial}^i + \bar{N}_{\alpha\beta} \dot{\partial}^\beta \end{cases} \quad (13)$$

which make the warped horizontal distribution on T^*M denoted by HT^*M . The dual 1-forms of these local vector fields on T^*T^*M are defined by followings

$$\begin{cases} d\mathbf{x}^a = dx^i \delta_i^a + dz^\alpha \delta_{\alpha+n}^a \\ \delta^* p_i := \delta \mathbf{p}_i = dp_i - \bar{N}_{ij} dx^j - \bar{N}_{i\alpha} dz^\alpha \\ \delta^* q_\alpha := \delta \mathbf{p}_{\alpha+n} = dq_\alpha - \bar{N}_{\alpha i} dx^i - \bar{N}_{\alpha\beta} dz^\beta \end{cases} \quad (14)$$

The Sasakian lift of metric tensor (10) on the local frame (14) is expressed by

$$G = g_{ij} dx^i \otimes dx^j + \frac{g_{\alpha\beta}}{f} dz^\alpha \otimes dz^\beta + g^{ij} \delta^* p_i \otimes \delta^* p_j + f g^{\alpha\beta} \delta^* q_\alpha \otimes \delta^* q_\beta \quad (15)$$

4 The Levi-Civita Connection of Metric G

The Lie brackets of the local vector fields given in previous section are presented as follows

$$\begin{cases} [\frac{\delta^*}{\delta^* x^i}, \frac{\delta^*}{\delta^* x^j}] = \mathbf{R}_{ijk} \dot{\partial}^k + \mathbf{R}_{ij\alpha} \dot{\partial}^\alpha \\ [\frac{\delta^*}{\delta^* x^i}, \frac{\delta^*}{\delta^* z^\alpha}] = \mathbf{R}_{i\alpha j} \dot{\partial}^j + \mathbf{R}_{i\alpha\beta} \dot{\partial}^\beta \\ [\frac{\delta^*}{\delta^* z^\alpha}, \frac{\delta^*}{\delta^* z^\beta}] = \mathbf{R}_{\alpha\beta i} \dot{\partial}^i + \mathbf{R}_{\alpha\beta\gamma} \dot{\partial}^\gamma \end{cases} \quad (16)$$

where

$$\begin{cases} \mathbf{R}_{ijk} = \frac{\delta^* \bar{N}_{jk}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{ik}}{\delta^* x^j}, & \mathbf{R}_{ij\alpha} = \frac{\delta^* \bar{N}_{j\alpha}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{i\alpha}}{\delta^* x^j} \\ \mathbf{R}_{i\alpha k} = \frac{\delta^* \bar{N}_{\alpha k}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{ik}}{\delta^* z^\alpha}, & \mathbf{R}_{i\alpha\beta} = \frac{\delta^* \bar{N}_{\alpha\beta}}{\delta^* x^i} - \frac{\delta^* \bar{N}_{i\beta}}{\delta^* z^\alpha} \\ \mathbf{R}_{\alpha\beta k} = \frac{\delta^* \bar{N}_{\beta k}}{\delta^* z^\alpha} - \frac{\delta^* \bar{N}_{\alpha k}}{\delta^* z^\beta}, & \mathbf{R}_{\alpha\beta\gamma} = \frac{\delta^* \bar{N}_{\beta\gamma}}{\delta^* z^\alpha} - \frac{\delta^* \bar{N}_{\alpha\gamma}}{\delta^* z^\beta} \end{cases} \quad (17)$$

The components \mathbf{R}_{abc} are called *curvature tensors* of nonlinear connection \bar{N}_{ab} and they are skew-symmetric with respect to the indices a and b . Moreover

$$\left\{ \begin{array}{l} [\dot{\partial}^i, \frac{\delta^*}{\delta^* x^j}] = \dot{\partial}^i(\bar{N}_{jk})\dot{\partial}^k \\ [\dot{\partial}^\alpha, \frac{\delta^*}{\delta^* x^i}] = \dot{\partial}^\alpha(\bar{N}_{ik})\dot{\partial}^k + \dot{\partial}^\alpha(\bar{N}_{i\beta})\dot{\partial}^\beta \\ [\dot{\partial}^i, \frac{\delta^*}{\delta^* z^\alpha}] = \dot{\partial}^i(\bar{N}_{\alpha\beta})\dot{\partial}^\beta \\ [\dot{\partial}^\alpha, \frac{\delta^*}{\delta^* z^\beta}] = \dot{\partial}^\alpha(\bar{N}_{\beta k})\dot{\partial}^k + \dot{\partial}^\alpha(\bar{N}_{\beta\gamma})\dot{\partial}^\gamma \end{array} \right. \quad (18)$$

Let ∇ be the Levi-Civita connection on (T^*M, G) which is given by:

$$\left\{ \begin{array}{l} 2G(\nabla_X Y, Z) = XG(Y, Z) + YG(X, Z) - ZG(X, Y) \\ -G([X, Z], Y) - G([Y, Z], X) + G([X, Y], Z) \end{array} \right. \quad (19)$$

for any $X, Y, Z \in \Gamma(TT^*M)$. Then, the components of ∇ are calculated as follows:

$$\begin{aligned} \nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j} &= \Gamma_{ij}^k \frac{\delta^*}{\delta^* x^k} - \frac{f}{2} \bar{N}_{\alpha k} g_{ij}^k g^{\alpha\beta} \frac{\delta^*}{\delta^* z^\beta} + \frac{1}{2} g_{ijk} \dot{\partial}^k + \frac{1}{2} \mathbf{R}_{ija} \bar{\partial}^a \\ \nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* z^\alpha} &= \nabla_{\frac{\delta^*}{\delta^* z^\alpha}} \frac{\delta^*}{\delta^* x^i} + \mathbf{R}_{i\alpha a} \bar{\partial}^a = -\frac{1}{2} \bar{N}_{\alpha j} g_i^{jk} \frac{\delta^*}{\delta^* x^k} \\ &+ \frac{1}{2} \left(\frac{\partial \ln f}{\partial x^i} \delta_\alpha^\gamma - \bar{N}_{i\beta} g_\alpha^{\beta\gamma} \right) \frac{\delta^*}{\delta^* z^\gamma} + \frac{1}{2} \mathbf{R}_{i\alpha a} \bar{\partial}^a \\ \nabla_{\frac{\delta^*}{\delta^* z^\alpha}} \frac{\delta^*}{\delta^* z^\beta} &= -\frac{1}{2} \frac{\delta^* f g_{\alpha\beta}}{\delta^* x^i} g^{ij} \frac{\delta^*}{\delta^* x^j} + \Gamma_{\alpha\beta}^\gamma \frac{\delta^*}{\delta^* z^\gamma} + \frac{1}{2} f^2 g_{\alpha\beta\lambda} \dot{\partial}^\lambda \\ &+ \frac{1}{2} \mathbf{R}_{\alpha\beta a} \bar{\partial}^a \\ \nabla_{\dot{\partial}^i} \dot{\partial}^\alpha &= \nabla_{\dot{\partial}^\alpha} \dot{\partial}^i = \frac{1}{8} \dot{\partial}^\alpha H_2 g^{ikh} \frac{\partial f}{\partial x^k} \frac{\delta^*}{\delta^* x^h} \\ &- \frac{1}{2} (f^2 \dot{\partial}^i (\bar{N}_{\beta\gamma}) g^{\gamma\alpha} g^{\beta\lambda} + f \dot{\partial}^\alpha (\bar{N}_{\beta k}) g^{ki} g^{\beta\lambda}) \frac{\delta^*}{\delta^* z^\lambda} \\ \nabla_{\dot{\partial}^i} \dot{\partial}^j &= -\frac{1}{2} \left(\frac{\delta^* g^{ij}}{\delta^* x^k} + \dot{\partial}^i (\bar{N}_{kt}) g^{tj} + \dot{\partial}^j (\bar{N}_{kt}) g^{ti} \right) g^{kh} \frac{\delta^*}{\delta^* x^h} \\ &+ \frac{1}{8} \dot{\partial}^\beta H_2 g^{ijk} \frac{\partial f}{\partial x^k} \frac{\delta^*}{\delta^* z^\beta} + \frac{1}{2} g_k^{ij} \dot{\partial}^k \\ \nabla_{\dot{\partial}^\alpha} \dot{\partial}^\beta &= -\frac{1}{2} \left(\frac{\delta^* f g^{\alpha\beta}}{\delta^* x^k} + f \dot{\partial}^\alpha (\bar{N}_{k\gamma}) g^{\gamma\beta} + f \dot{\partial}^\beta (\bar{N}_{k\gamma}) g^{\gamma\alpha} \right) g^{kh} \frac{\delta^*}{\delta^* x^h} \\ &- \frac{f^2}{2} \left(\frac{\delta^* g^{\alpha\beta}}{\delta^* z^\gamma} + \dot{\partial}^\alpha (\bar{N}_{\gamma\theta}) g^{\theta\beta} + \dot{\partial}^\beta (\bar{N}_{\gamma\theta}) g^{\theta\alpha} \right) g^{\gamma\lambda} \frac{\delta^*}{\delta^* z^\lambda} + \frac{1}{2} g_\gamma^{\alpha\beta} \dot{\partial}^\gamma \end{aligned} \quad (21)$$

$$\begin{aligned}
\nabla_{\frac{\delta^*}{\delta^* x^i}} \dot{\partial}^j &= \nabla_{\dot{\partial}^j} \frac{\delta^*}{\delta^* x^i} - \dot{\partial}^j (\bar{N}_{ik}) \dot{\partial}^k = -\frac{1}{2} \dot{\partial}^j (\bar{N}_{ik}) \dot{\partial}^k \\
&\quad - \frac{1}{2} (g_i^{jh} + \mathbf{R}_{iks} g^{sj} g^{kh}) \frac{\delta^*}{\delta^* x^h} - \frac{f}{2} \mathbf{R}_{iak} g^{kj} g^{\alpha\beta} \frac{\delta^*}{\delta^* z^\beta} \\
&\quad + \frac{1}{2} \left(\frac{\delta^* g^{jk}}{\delta^* x^i} + \dot{\partial}^k (\bar{N}_{is}) g^{sj} \right) g_{kh} \dot{\partial}^h + \frac{1}{2f} \dot{\partial}^\alpha (\bar{N}_{ik}) g^{kj} g_{\alpha\beta} \dot{\partial}^\beta \\
\nabla_{\frac{\delta^*}{\delta^* x^i}} \dot{\partial}^\alpha &= \nabla_{\dot{\partial}^\alpha} \frac{\delta^*}{\delta^* x^i} - \dot{\partial}^\alpha (\bar{N}_{ia}) \bar{\partial}^a = -\frac{1}{2} \dot{\partial}^\alpha (\bar{N}_{ia}) \bar{\partial}^a \\
&\quad + \frac{f}{2} \mathbf{R}_{kib} g^{\beta\alpha} g^{kh} \frac{\delta^*}{\delta^* x^h} + \frac{f^2}{2} \mathbf{R}_{\beta i \gamma} g^{\gamma\alpha} g^{\beta\lambda} \frac{\delta^*}{\delta^* z^\lambda} \\
&\quad + \frac{1}{2} \left(\frac{1}{f} \frac{\delta^* f g^{\alpha\beta}}{\delta^* x^i} + \dot{\partial}^\beta (\bar{N}_{i\gamma}) g^{\gamma\alpha} \right) g_{\beta\lambda} \dot{\partial}^\lambda \\
\nabla_{\frac{\delta^*}{\delta^* z^\alpha}} \dot{\partial}^i &= \nabla_{\dot{\partial}^i} \frac{\delta^*}{\delta^* z^\alpha} - \dot{\partial}^i (\bar{N}_{\alpha\beta}) \dot{\partial}^\beta = -\frac{1}{2} \dot{\partial}^i (\bar{N}_{\alpha\beta}) \dot{\partial}^\beta \\
&\quad + \frac{1}{2} \mathbf{R}_{k\alpha s} g^{si} g^{kh} \frac{\delta^*}{\delta^* x^h} + \frac{f}{2} \mathbf{R}_{\beta\alpha k} g^{ki} g^{\beta\gamma} \frac{\delta^*}{\delta^* z^\gamma} + \frac{1}{2} \frac{\delta^* g^{ik}}{\delta^* z^\alpha} g_{kh} \dot{\partial}^h \\
&\quad + \frac{1}{2f} \dot{\partial}^\beta (\bar{N}_{\alpha k}) g^{ki} g_{\beta\gamma} \dot{\partial}^\gamma \\
\nabla_{\frac{\delta^*}{\delta^* z^\alpha}} \dot{\partial}^\beta &= \nabla_{\dot{\partial}^\beta} \frac{\delta^*}{\delta^* z^\alpha} - \dot{\partial}^\beta (\bar{N}_{\alpha a}) \bar{\partial}^a = -\frac{1}{2} \dot{\partial}^\beta (\bar{N}_{\alpha a}) \bar{\partial}^a \\
&\quad + \frac{f}{2} \mathbf{R}_{k\alpha\gamma} g^{\gamma\beta} g^{kh} \frac{\delta^*}{\delta^* x^h} - \frac{1}{2} (g_\alpha^{\beta\lambda} + f^2 \mathbf{R}_{\alpha\gamma\theta} g^{\theta\beta} g^{\gamma\lambda}) \frac{\delta^*}{\delta^* z^\lambda} \\
&\quad - \frac{1}{4f} \delta_\alpha^\beta \frac{\partial f}{\partial x^j} \dot{\partial}^j + \frac{1}{2} \left(\frac{\delta^* g^{\beta\gamma}}{\delta^* z^\alpha} g_{\gamma\lambda} + \dot{\partial}^\gamma (\bar{N}_{\alpha\theta}) g^{\theta\beta} g_{\gamma\lambda} \right) \dot{\partial}^\lambda
\end{aligned} \tag{22}$$

where,

$$g^{abc} = \bar{\partial}^a g^{bc}, \quad g_{abc} = g_{cf} g_{ab}^f = g_{cf} g_{be} g_a^{ef} = g_{cf} g_{be} g_{ad} g^{def}$$

and

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{g^{kh}}{2} \left(\frac{\delta^* g_{jh}}{\delta^* x^i} + \frac{\delta^* g_{ih}}{\delta^* x^j} - \frac{\delta^* g_{ij}}{\delta^* x^h} \right), \\
\Gamma_{\alpha\beta}^\gamma &= \frac{g^{\gamma\lambda}}{2} \left(\frac{\delta^* g_{\beta\lambda}}{\delta^* z^\alpha} + \frac{\delta^* g_{\alpha\lambda}}{\delta^* z^\beta} - \frac{\delta^* g_{\alpha\beta}}{\delta^* z^\lambda} \right).
\end{aligned}$$

5 Foliations on Warped Product Hamiltonian Spaces

In this section, we study the geometric properties of vertical distribution VT^*M such as being bundle-like with respect to the metric G and being totally geodesic. The conditions which are equivalent to these properties show a close relation between the geometry of warped Hamilton manifold and its base Hamiltonian spaces. Our results are presented as follows

Theorem 5.1. *Let $\mathbb{H} = (M, H)$ be a warped product Hamilton space with nonconstant warped function f . Then, the warped sasaki metric G (15) is bundlelike for vertical foliation VT^*M if and only if $(M_1, (g_{ij}))$ and $(M_2, (g_{\alpha\beta}))$ are two Riemannian manifolds.*

Proof. With respect to bundle-like condition (see [4, 14]), G is bundle-like for VT^*M if and only if:

$$G(\nabla_X Y + \nabla_Y X, Z) = 0 \quad \forall X, Y \in \Gamma(HT^*M), Z \in \Gamma(VT^*M)$$

where, it is equivalent to followings equations:

$$\begin{aligned} G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j} + \nabla_{\frac{\delta^*}{\delta^* x^j}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^k) &= G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j} + \nabla_{\frac{\delta^*}{\delta^* x^j}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^\alpha) = 0 \\ G(\nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* u^\beta} + \nabla_{\frac{\delta^*}{\delta^* u^\beta}} \frac{\delta^*}{\delta^* u^\alpha}, \dot{\partial}^i) &= G(\nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* u^\beta} + \nabla_{\frac{\delta^*}{\delta^* u^\beta}} \frac{\delta^*}{\delta^* u^\alpha}, \dot{\partial}^\gamma) = 0 \\ G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* u^\alpha} + \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^j) &= G(\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* u^\alpha} + \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* x^i}, \dot{\partial}^\beta) = 0 \end{aligned}$$

Therefore, by (22) it is obtained that $g_{ijk} = g_{\alpha\beta\gamma} = 0$ and these complete the proof. \square

Theorem 5.2. *Let $\mathbb{H} = (M, H)$ be a warped product Hamilton space with nonconstant warped function f . Then, $\mathbb{H} = (M, H)$ is a Landsberg-Hamilton space if and only if the vertical foliation VT^*M is totally geodesic.*

Proof. First, we let

$$g_{ab|*c} = \frac{\delta^* g_{ab}}{\delta^* x^c} + g^{bd} \dot{\partial}^a (\bar{N}_{dc}) + g^{ad} \dot{\partial}^b (\bar{N}_{dc})$$

Now, (M, H) is a Landsberg-Hamilton space if and only if

$$g_{ab|*c} = 0$$

then, by means of Eq. (22) the last equation is equivalent to the totally geodesic properties of VT^*M . \square

Theorem 5.3. *Let $\mathbb{H} = (M, H)$ be a warped product Hamilton space with nonconstant warped function f . Then, the horizontal distribution HT^*M is a totally geodesic one if and only if (M_1, H_1) and (M_2, H_2) are Euclidean spaces.*

Proof. Suppose that HT^*M is a totally geodesic distribution, then

$$\nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* x^j}, \nabla_{\frac{\delta^*}{\delta^* x^i}} \frac{\delta^*}{\delta^* u^\alpha}, \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* x^i}, \nabla_{\frac{\delta^*}{\delta^* u^\alpha}} \frac{\delta^*}{\delta^* u^\beta} \in \Gamma(HT^*M)$$

From Eq. (22), these are equivalent to

$$\mathbf{R}_{abc} = g_{abc} = 0$$

and these show (M, H) is a Riemannian Flat manifold or an Euclidean one. The inverse is true as well. \square

Gathering the theorems 5.1 and 5.2, we have the following corollary.

Corollary 5.1. *Consider warped product Hamilton space (M, H) be a Riemannian manifold with nonconstant warped function f , then the vertical distribution VT^*M is totally geodesic and metric G is bundle-like for VT^*M .*

References

- [1] Y. Alipour-Fakhri and M. M. Rezaei, *The warped Sasaki-Matsumoto metric and bundlelike condition*, Journal of Mathematical Physics, 51 (2010), 122701-1~122701-13.
- [2] D. Bao, S.S. Chen and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, New York, 2000.
- [3] A. Bejancu, *Tangent Bundle and Indicatrix Bundle of a Finsler Manifold*, Journal of Kodai Mathematics, 31 (2008), 272-306.
- [4] A. Bejancu and H. R. Farran, *Foliations and Geometric Structures*, Springer-Verlag, Netherlands, 2006.
- [5] A. Bejancu, *Oblique Warped Products*, Journal of Geometry and Physics, 57 (2007), 1055-1073.
- [6] R. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., 46 (1969), 1-49.
- [7] A. B. Hushmandi and M. M. Rezaei, *On warped product Finsler spaces of Landsberg type*, 52, 093506 (2011).
- [8] B. H. Kim, *Warped Product Spaces With Einstein Metric*, Comm. Korean Math. Soc., 8 (1993), 467-473.
- [9] L. Kozma and I. R. Peter and C. Varga, *Warped product of Finsler manifolds*, Ann. Univ. Sci. Pudapest, 44 (2001), 157-170.
- [10] R. Miron and M. Anastasiei, *Vector Bundles and Lagrange Spaces with Applications to Relativity*, Geometry Balkan Press, no.1, 1997.
- [11] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., FTPH, no.59, 1994.
- [12] R. Miron, D. Hrimiuc, H. Shimada and S.V. Sabau, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, New York, 2002.

- [13] R. Miron, *Hamilton Geometry*, An. Șt. Al.I.Cuza Univ., Iași, s. I-a Mat., 35 (1989), 33-67.
- [14] P. Molino, *Riemannian Foliations*, Progress in Math., Birkhauser, Boston, 1988.
- [15] B. O'Neill, *Semi-riemannian geometry with application to relativity*, Academic Press, New York, 1983.
- [16] B. L. Reinhart, *Foliated manifolds with bundle-like metric*, Annals of Math., 69 (1959), 119-132.
- [17] M. M. Rezaii and Y. Alipour-fakhri, *On Projectively Related Warped Product Finsler Manifolds*, 8 (2011), 953-967.
- [18] Ph. Tondeur, *Geometry of Foliations*, Birkhäuser, Basel, 1997.

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